Euler's method

1. Use four steps of Euler's method to approximate a solution on the interval [0, 1] to the initial-value problem defined by

$$y^{(1)}(t) = -y(t) + t - 2$$
$$y(0) = 1$$

Answer: 1.0, 0.250, –0.250, –0.56250, –0.7343750

2. Use eight steps of Euler's method to approximate a solution on the interval [0, 1] to the initial-value problem defined that shown in Question 1.

Answer: To ten digits of significance, 1, 0.625, 0.3125, 0.0546875, –0.1552734375, –0.3233642578, –0.4548187256, –0.5542163849, –0.6255643368.

3. If the actual solution is $y(t) = 4e^{-t} + t - 3$, argue that this method is indeed $O(h^2)$ for a single step.

Answer: To four significant digits, the error of the approximation of $y(0.25)$ in Question 1 is 0.1152 and the error of the approximation of $y(0.125)$ in Question 2 is 0.02999, and this second value is approximately one quarter the error of the first.

4. If the actual solution is $y(t) = 4e^{-t} + t - 3$, argue that this method is indeed $O(h)$ over multiple steps.

Answer: $y(1) = 4e^{-1} + 1 - 3 \approx -0.5284822353142307136$, so the error of the approximation in Question 1 is approximately 0.2059 while the error with the second approximation is 0.09708, which is approximately half that of the previous approximation.

5. Note that the 2nd-derivative of the solution to the ivp in Question 1 is $4e^{-t}$, and thus the second derivative on the range [0, 1] goes from $[4e^{-1}, 4]$. Does this make sense with respect to the error analysis?

Answer: The error would be ½ multiplied by $t_f - t_0$ multiplied by h and the value of the second deritvative evaluated somewhere on the interval $[t_0, t_f]$. Thus, in this case, the errors should lie on the intervals [0.1839, 0.5] and [0.09197, 0.25], and in both cases, the error falls in the given interval.